

A CORRECTION AND SOME ADDITIONS TO “FUNDAMENTAL SOLUTIONS FOR DIFFERENTIAL EQUATIONS ASSOCIATED WITH THE NUMBER OPERATOR”

BY

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ABSTRACT. Let (H, B) be an abstract Wiener pair and $\mathcal{N}$ the operator defined by $\mathcal{N}u(x) = -\text{trace}_H D^2u(x) + (x, Du(x))$, where $x \in B$ and $(\cdot, \cdot)$ denotes the B - B^* pairing. In this paper, we point out a mistake in the previous paper concerning the existence of fundamental solutions of $\mathcal{N}^k$ and intend to make a correction. For this purpose, we study the fundamental solution of the operator $(\mathcal{N} + \lambda I)^k$ ($\lambda > 0$) and investigate its behavior as $\lambda \rightarrow 0$. We show that there exists a family $\{Q_\lambda(x, dy)\}$ of measures which serves as the fundamental solution of $(\mathcal{N} + \lambda I)^k$ and, for a suitable function f , we prove that the solution of $\mathcal{N}^k u = f$ can be represented by $u(x) = \lim_{\lambda \rightarrow 0} \int_B f(y) Q_\lambda(x, dy) + C$, where C is a constant.

In our previous paper [2, §3], we have shown that the solution of the equation $\mathcal{N}^k u(x) f(x)$ ($f \in \mathcal{L}_0$) is of the form $G^k f(x) + \text{a constant}$, where $Gf(x) = \int_0^\infty [\int_B f(y) o_t(x, dy)] dt$ and $G^k f = G(G^{k-1} f)$ with $G^0 f = f$. Viewing the representation of $G^k f$, we then intuitively claimed that the family $\{Q(x, dy)\}$ of k -fold convolution of $G(x, dy) = \int_0^\infty o_t(x, dy) dt$ forms rigorously the “fundamental solution” of $\mathcal{N}^k$. Unfortunately, the “fundamental solution” is only formal. The mistake is caused by the fact that $G^k f(x)$ may not equal $\int_B f(y) Q(x, dy)$ when $f \in \mathcal{L}_0$ (though $G^k f(x) = \int_B f(y) Q(x, dy)$ for all $f \geq 0$). In order to obtain a correct representation of $G^k f(x)$ by an integral with respect to certain measure, we study the fundamental solution of the differential operator $(\mathcal{N} + \lambda I)^k$, where $\lambda > 0$, and then investigate its behavior as λ goes to zero. We show that the fundamental solution of $(\mathcal{N} + \lambda I)^k$ exists in the sense of measure, which means that there exists a family of measures, say $\{Q_\lambda(x, dy)\}$, so that, for any member f of a certain reasonable large class of functions, the integral $Q_\lambda f(x) = \int_B f(y) Q_\lambda(x, dy)$ exists and $(\mathcal{N} + \lambda I)^k(Q_\lambda f)(x) = f(x)$. As λ goes to zero, we show that $\lim_{\lambda \rightarrow 0} \int_B f(y) Q_\lambda(x, dy) = G^k f(x)$ for any f in $\mathcal{L}_0$.

DEFINITIONS AND NOTATION. We give in the following some new definitions and notations which did not appear in the previous paper. For the others, we refer the reader to [2].

For each x in B and for each Borel set A in B , we define

$$G_\lambda(x, A) = \int_0^\infty e^{-\lambda t} o_t(x, A) dt \quad (\lambda > 0),$$

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$$R_\lambda(x, A) = \int_0^\infty e^{-\lambda t} [o_t(x, A) - p_1(A)] dt;$$

and let

$$G_\lambda f(x) = \int_0^\infty \int_B e^{-\lambda t} f(y) o_t(x, dy) dt \quad (\text{if it exists}),$$

$$R_\lambda f(x) = \int_0^\infty \int_B e^{-\lambda t} f(y) [o_t(x, dy) - p_1(dy)] dt.$$

Evidently, $G_\lambda f$ and $R_\lambda f$ exist when f is bounded and continuous. Furthermore, we have

LEMMA 1. (a) $G_\lambda(x, \cdot)$ and $R_\lambda(x, \cdot)$ are Borel measures with total variation λ^{-1} and $2\lambda^{-1}$, respectively.

(b) If $f \in \mathcal{L}$, then $R_\lambda f \in \mathcal{L}$ and $G_\lambda f \in \mathcal{L}$; and, if $f \in \mathcal{L}_0$, then $R_\lambda f(x) = G_\lambda f(x)$ and $G_\lambda f \in \mathcal{L}_0$.

(c) If $f \in \mathcal{L}$, f is integrable with respect to $R_\lambda(x, \cdot)$ and $G_\lambda(x, \cdot)$. Moreover, we have:

$$(1) \quad R_\lambda f(x) = \int_B f(y) R_\lambda(x, dy),$$

$$(2) \quad G_\lambda f(x) = \int_B f(y) G_\lambda(x, dy).$$

PROOF. (a) follows from the fact that $o_t(x, \cdot)$ and $p_1(\cdot)$ are mutually singular probability measures.

(b) follows by arguments similar to [2, Proposition 3.1].

It remains to prove (c). First of all, we observe that $R_\lambda f(x) = G_\lambda f(x) - \lambda^{-1} \int_B f(y) p_1(dy)$ and $R_\lambda(x, \cdot) = G_\lambda(x, \cdot) - \lambda^{-1} p_1(\cdot)$, so it suffices to verify (2).

Next, noting that if f is in $\mathcal{L}$ then f^+ , f^- and $|f|$ are also in $\mathcal{L}$; it suffices to prove that any nonnegative member f in $\mathcal{L}$ is integrable with respect to $G_\lambda(x, \cdot)$ and (2) holds. But, by the definition of $G_\lambda(x, \cdot)$, it is easy to see that (2) holds when f is a simple function and so, by the monotone convergence theorem, (2) holds if f is a nonnegative function. Now the integrability of a nonnegative member in $\mathcal{L}$ follows immediately from (b). $\square$

PROPOSITION 1. For each x in B and each Borel set E in B , define

$$(3) \quad Q_\lambda(x, E) = \int_B \cdots \int_B G_\lambda(y_{k-1}, E) G_\lambda(y_{k-2}, dy_{k-1}) \cdots G_\lambda(y_1, dy_2) G_\lambda(x, dy_1). \\ (k-1 \text{ times})$$

We have:

(a) The total variation of $Q_\lambda(x, \cdot)$ is λ^{-k} .

(b) $\mathcal{L} \subset L^1(Q_\lambda(x, \cdot))$ for each x in B and $\lambda > 0$ and

$$(4) \quad G_\lambda^k f(x) = \int_B f(y) Q_\lambda(x, dy).$$

(c) If f is a function in $\mathcal{L}$, then $u(x) = G_\lambda^k f(x)$ satisfies the equation $(\mathcal{Q} + \lambda I)^k u = f$ (cf. [1]).

PROOF. (a) follows from Lemma 1(a).

(b) Using the same idea as in the proof of Lemma 1(c), we see that f^+ and f^- are integrable with respect to $Q_\lambda(x, dy)$ and $G_\lambda^k f^+(x) = \int_B f^+(y) Q_\lambda(x, dy)$, $G_\lambda^k f^-(x) = \int_B f^-(y) Q_\lambda(x, dy)$, which yield the identity (4).

Finally, imitating the proof of [2, Theorem 3.5], (c) follows immediately. $\square$

REMARK. Proposition 1 shows that the fundamental solution of $(\mathcal{U} + \lambda I)^k$ exists in the sense of measure which is given by the family $\{Q_\lambda(x, \cdot)\}$. $\square$

PROPOSITION 2. Let $\{f_\lambda: \lambda \in R^+\}$ be a net of functions in $\mathcal{L}$ satisfying the following conditions:

(C-1) There exist constants c, c' such that

$$|f_\lambda(x) - f_\lambda(y)| \leq c \cdot e^{c'\|x\|} e^{c'\|y\|} \|x - y\|$$

for all $x, y \in B$ and $\lambda \in R^+$.

(C-2) $\lim_{\lambda \rightarrow 0} f_\lambda(x) = f(x)$.

Then we have

$$(5) \quad \lim_{\lambda \rightarrow 0} R_\lambda f_\lambda(x) = \int_0^\infty [o_t f(x) - p_1 f(0)] dt.$$

In particular, if $f \in \mathcal{L}$, then $\lim_{\lambda \rightarrow 0} R_\lambda f(x) = Rf(x)$, where $Rf(x)$ is defined by the limit function of (5).

PROOF. Write out the expression of $R_\lambda f_\lambda(x)$ and use Lebesgue's dominated convergence theorem. $\square$

COROLLARY 1. Assume $f \in \mathcal{L}_0$. Then

$$G^k f(x) = \lim_{\lambda \rightarrow 0} \int_B f(y) Q_\lambda(x, dy).$$

PROOF. Noting that the net $\{G_\lambda f\}$ satisfies (C-1) and (C-2) of Proposition 2, the Corollary follows immediately. $\square$

REMARK. To correct the previous paper, we should change properly all the statements concerning the fundamental solution of $\mathcal{U}^k$ according to the above results. In view of Corollary 1. Theorem 3.5(b) of [2] should read:

Assume f is a function in $\mathcal{L}_0$ and $Q_\lambda(x, \cdot)$ is defined as in (3). Then $G^k f(x) = \lim_{\lambda \rightarrow 0} \int_B f(y) Q_\lambda(x, dy)$ exists, $G^k f \in \mathcal{L}(k)_0$ and $\mathcal{U}^k(G^k f)(x) = f(x)$. $\square$

REMARK. It is not known so far if the fundamental solution of $\mathcal{U}^k$ exists in the sense of measure. When $k = 1$ and $f \in \mathcal{L}_0$, we see that $p_1 f(0) = 0$ and

$$Gf(x) = Rf(x) = \int_0^\infty (o_t f(x) - 0) dt = \int_0^\infty \int_B f(y) [o_t(x, dy) - p_1(dy)] dt.$$

Since the last integral exists for all $f \in \mathcal{L}$, one might conjecture that the the set function $R(x, A) = \int_0^\infty [o_t(x, A) - p_1(A)] dt$ could define a measure and the family $\{R(x, A)\}$ might form the fundamental solution of $\mathcal{U}$. Unfortunately, if one takes A = the concentrated set of p_1 , then $R(x, A) = -\infty$ and $R(x, A^c) = +\infty$, thus

$R(x, \cdot)$ fails to be a measure. From this observation, we conjecture that the fundamental solution of $\mathcal{U}$ does not exist in the sense of measure and neither does that of $\mathcal{U}^k$. However, a proof is lacking. $\square$

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